The orientable genus of some Cartesian products of bipartite graphs

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The drawing of graphs on surfaces knows its birth in the 19th century, mainly with the four colour problem and the Heawood map colouring problem. A surface is characterised by whether it is orientable or nonorientable and by its genus. The sphere is the orientable surface with genus 0 and is denoted by S_0. The genus g of an orientable surface S_g is the number of handles that are added to S_0, whereas the genus $\overline{g}$ of a nonorientable surface $N_{\overline{g}}$ is the number of Möbius bands (or crosscaps) that are added to S_0. For instance, the torus is the surface S_1 whereas the projective plane is the surface N_1.

A graph is embeddable on a surface if it can be drawn on that surface without edge crossings. There are two complementary perspectives that can be considered when embedding graphs on surfaces, namely fixing the graph and determining which surface allows an embedding of that graph, or fixing the surface and determining whether a graph is embeddable on that surface.

For a graph G, the (minimum) genus of G is the smallest genus of a surface which allows an embedding of G. More formally, a graph G has orientable genus g if it can be embedded on the orientable surface S_g but not on the orientable surface S_{g-1} (and similarly for the nonorientable genus of a graph). Determining the (orientable and nonorientable) genus of a graph is known to be NP-complete. A lot of work has been done to determine the genus of various families of graphs. Two of the initial results on these lines are given by Ringel and Youngs (1968) and by Ringel (1965), respectively determining the genus of the complete graphs and of the complete bipartite graphs. Since these initial results, other families of graphs were investigated for their genus, including the products of graphs belonging to the two aforementioned families.

In this talk we consider the Cartesian product of graphs since this is a well studied graph operation which is often used for modeling interconnection networks. This graph product provides a quick, effective and efficient way of constructing bigger graphs from smaller ones in such a way that the structure of the smaller graphs is preserved. The s-cube $Q_i^{(s)}$ is obtained by taking the repeated Cartesian product of i complete bipartite graphs $K_{s,s}$. We use a technique developed by Pisanski (1982) and White (1970), generally referred to as the White-Pisanski method, to determine the genus of the Cartesian product of the $2r$-cube with the repeated Cartesian product of cycles and of the Cartesian product of the $2r$-cube with the repeated Cartesian product of paths.